Advanced Digital Logic Design – EECS 303

http://ziyang.eecs.northwestern.edu/eecs303/

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Outline

1. Addition and subtraction
2. Multiplication
3. Homework

Section outline

1. Addition and subtraction
   - Overview
   - Number systems
   - Adders

Arithmetic/logic operations

- Increment
- Addition
- Negation
- Subtraction
- Multiplication
  - Slow or large
- Division
  - Slow or large

Arithmetic/logic operations

- Shift left
  - Fast, multiplication by two
- Shift right
  - Fast, division by two
- Bit-wise operations
  - AND, OR, NOT, NAND, NOR, XOR, and XNOR

Arithmetic

- Number systems review
- Adders
- Multipliers
- Memory overview

Arithmetic circuits

- Administration
- Number systems
- Adders
  - Ripple carry
  - Carry lookahead
  - Carry select

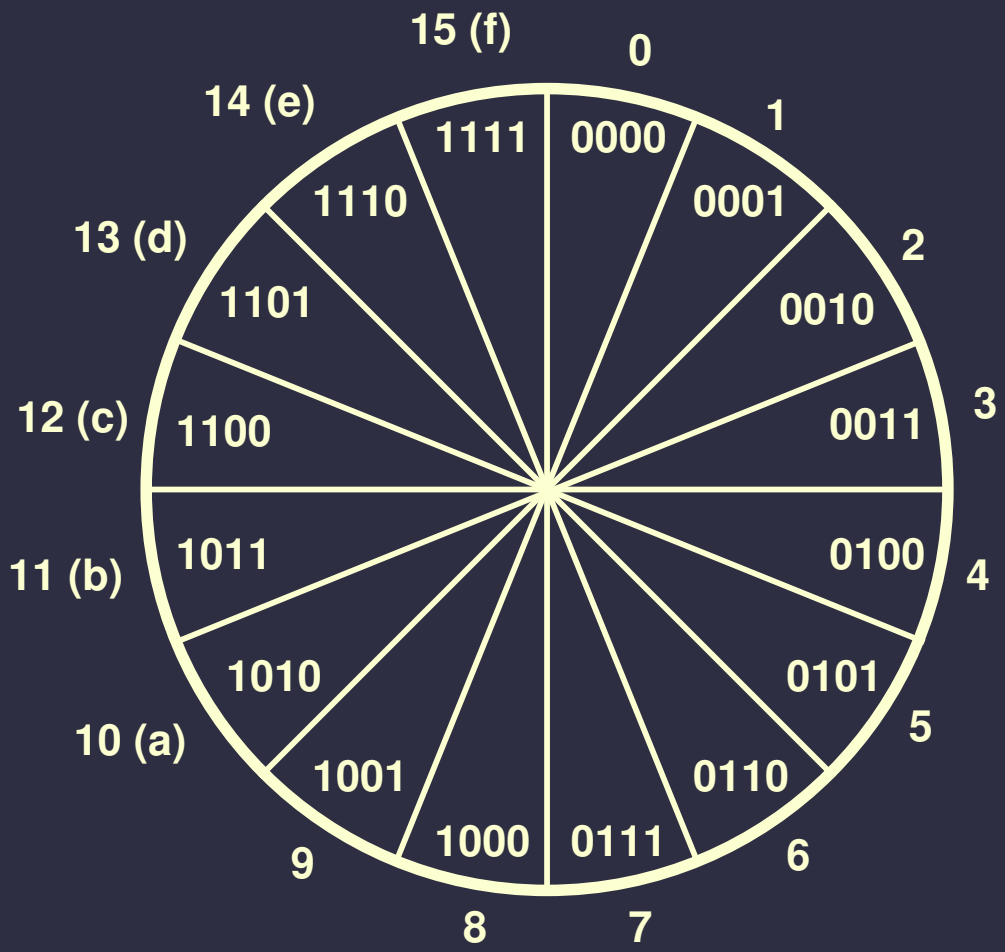
Section outline

1. Addition and subtraction
   - Overview
   - Number systems
   - Adders

Number systems

- Representation of positive numbers same in most systems
- A few special-purpose alternatives exist, e.g., Gray code
- Alternatives exist for signed numbers

Standard unsigned binary numbers



Given an n-bit number in which d_i is the ith digit, the number is $\sum_{i=1}^{n} 2^{i-1} d_i$

Unsigned addition

Consider adding 9 (1001) and 3 (0011)

$$\begin{array}{cccccc} & & 1 & 0 & 0 & 1 \\ + & & 0 & 0 & 1 & 1 \\ \hline \end{array}$$

Unsigned addition

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Why an extra column?

Overflow

- If the result of an operation can't be represented in the available number of bits, an *overflow* occurs
- E.g., $0110 + 1011 = 10001$
- Need to detect overflow

Overflow

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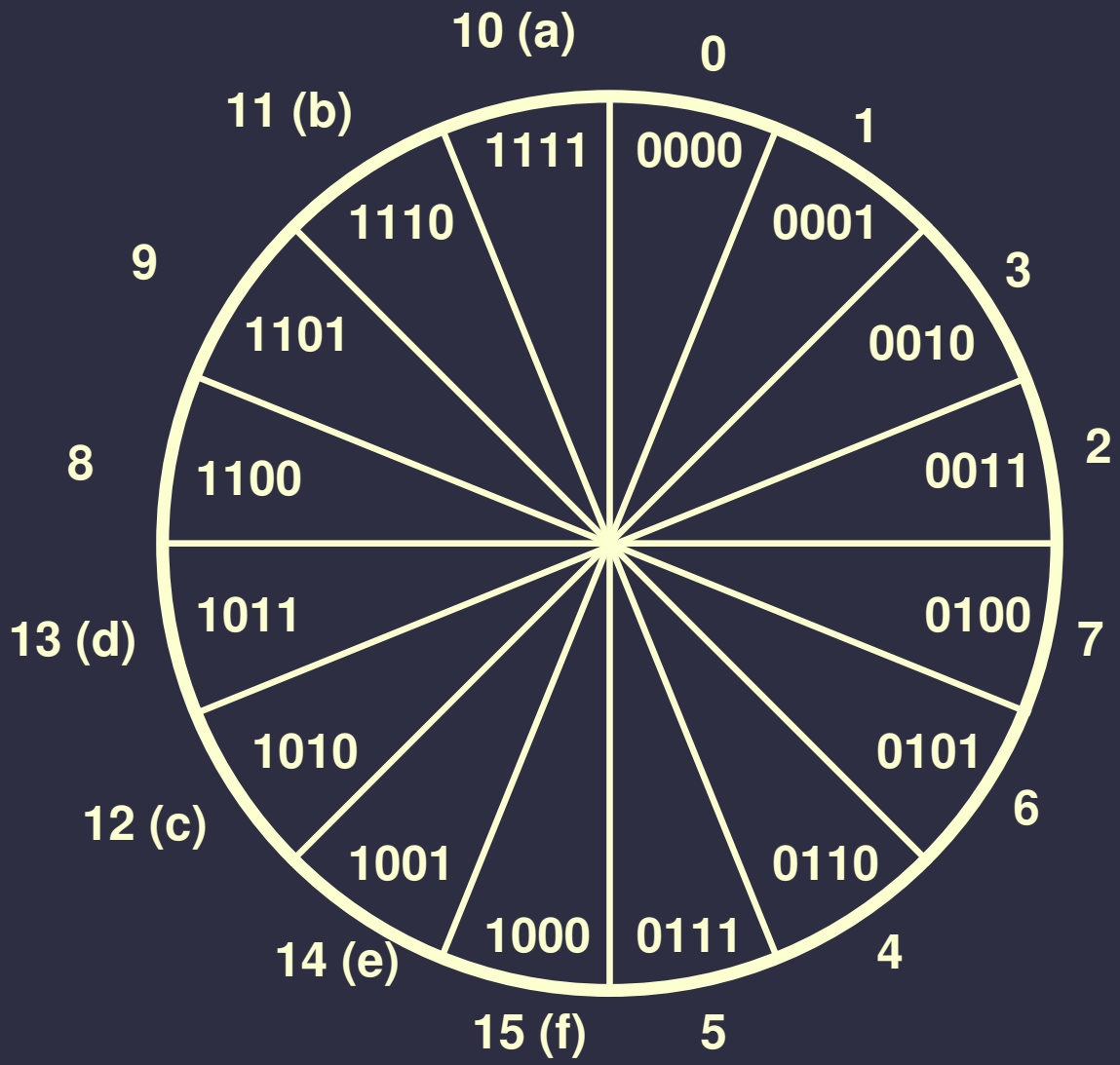
Gray code


10 (a)
0
11 (b)
1
9
3
8
2
13 (d)
7
12 (c)
6
14 (e)
4
15 (f)
5
1111
0000
1110
0001
1101
0010
1100
0011
1011
0100
1010
0101
1001
0110
1000
0111

Gray code

- To convert from a standard binary number to a Gray code number XOR the number by it's half (right-shift it)
- To convert from a Gray code number to a standard binary number, XOR each binary digit with the parity of the higher digits

Given that a number contains n digits and each digit, d_i, contributes 2^{i-1} to the number

$$\mathcal{P}_j^k = d_j \oplus d_{j+1} \cdots \oplus d_{k-1} \oplus d_k$$
$$d_i = d_i \oplus \mathcal{P}_{i+1}^n$$

Gray code

- Converting from Gray code to standard binary is difficult
  - Take time approximately proportional to n
- Doing standard arithmetic operations using Gray coded numbers is difficult
- Generally slower than using standard binary representation
- E.g., addition requires two carries
- Why use Gray coded numbers?
  - Analog to digital conversion
  - Reduced bus switching activity

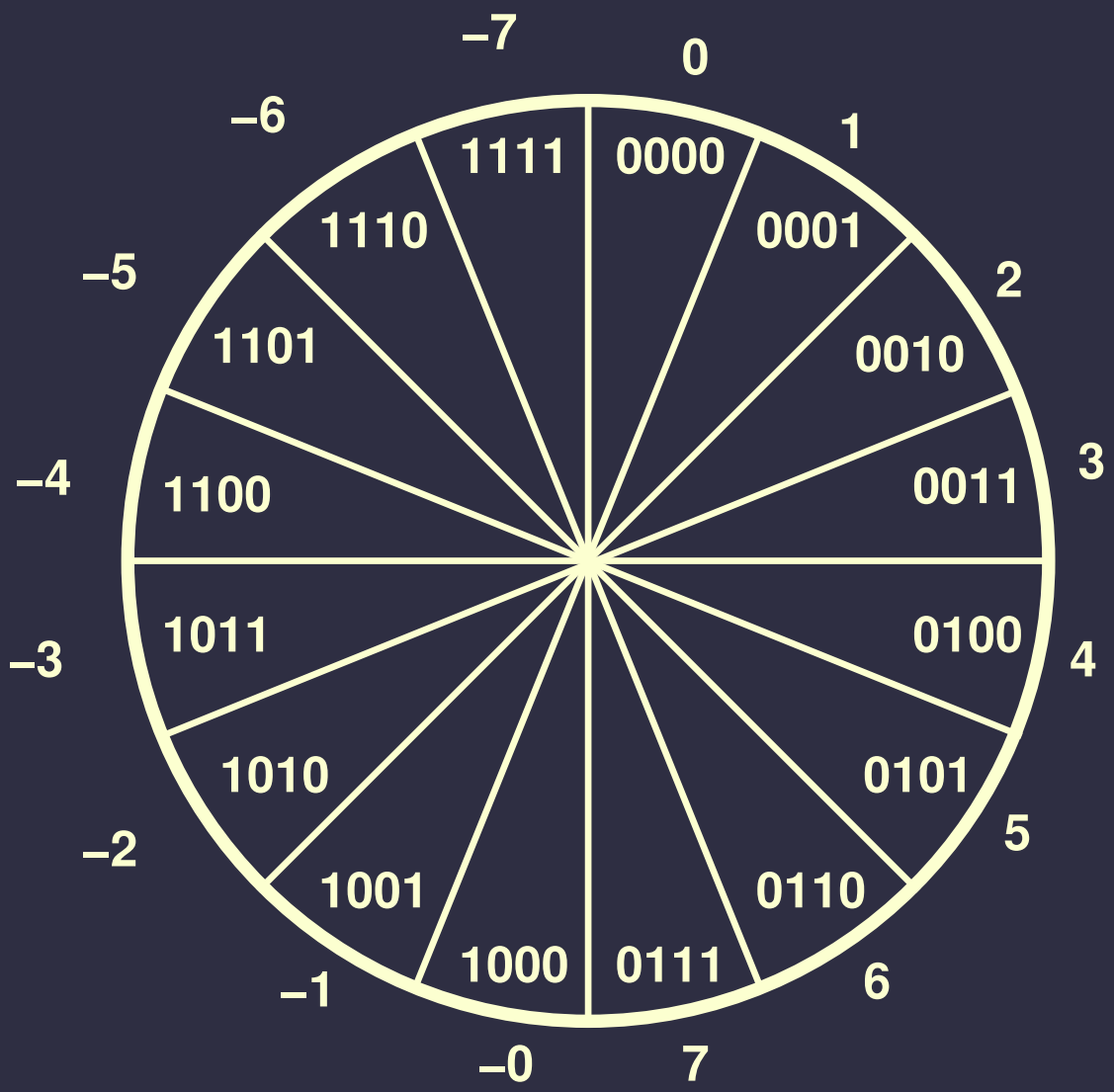
Signed number systems

- Three major schemes
  - Sign and magnitude
  - One's complement
  - Two's complement

Number system assumptions

- Four-bit machine word
- 16 values can be represented
- Approximately half are positive
- Approximately half are negative

Sign and magnitude



Sign and magnitude

- d_n represents sign
  - 0 is positive, 1 is negative
- Two representations for zero
- What is the range for such numbers?

Sign and magnitude

- d_n represents sign
  - 0 is positive, 1 is negative
- Two representations for zero
- What is the range for such numbers?
  - Range: $[-2^{n-1}+1, 2^{n-1}-1]$

Sign and magnitude

- How is addition done?
- If both numbers have the same sign, add them like unsigned numbers and preserve sign
- If numbers have differing signs, subtract smaller magnitude from larger magnitude and use sign of large magnitude number

Sign and magnitude

- Consider $5 + -6$
- Note that signs differ
- Use magnitude comparison to determine large magnitude: $6 - 5$
- Subtract smaller magnitude from larger magnitude: 1
- Use sign of large magnitude number: -1

Direct subtraction

Consider subtracting 5 (0101) from 6 (0110)

$$\begin{array}{rcccc} & 0 & 1 & 1 & 0 \\ - & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

- Note that this operation is different from addition
- Sign and magnitude addition is complicated

Direct subtraction

Consider subtracting 5 (0101) from 6 (0110)

$$\begin{array}{cccccc} & & & b & & \\ & 0 & 1 & 1 & 0 \\ - & 0 & 1 & 0 & 1 \\ \hline & & & & 1 \end{array}$$

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Direct subtraction

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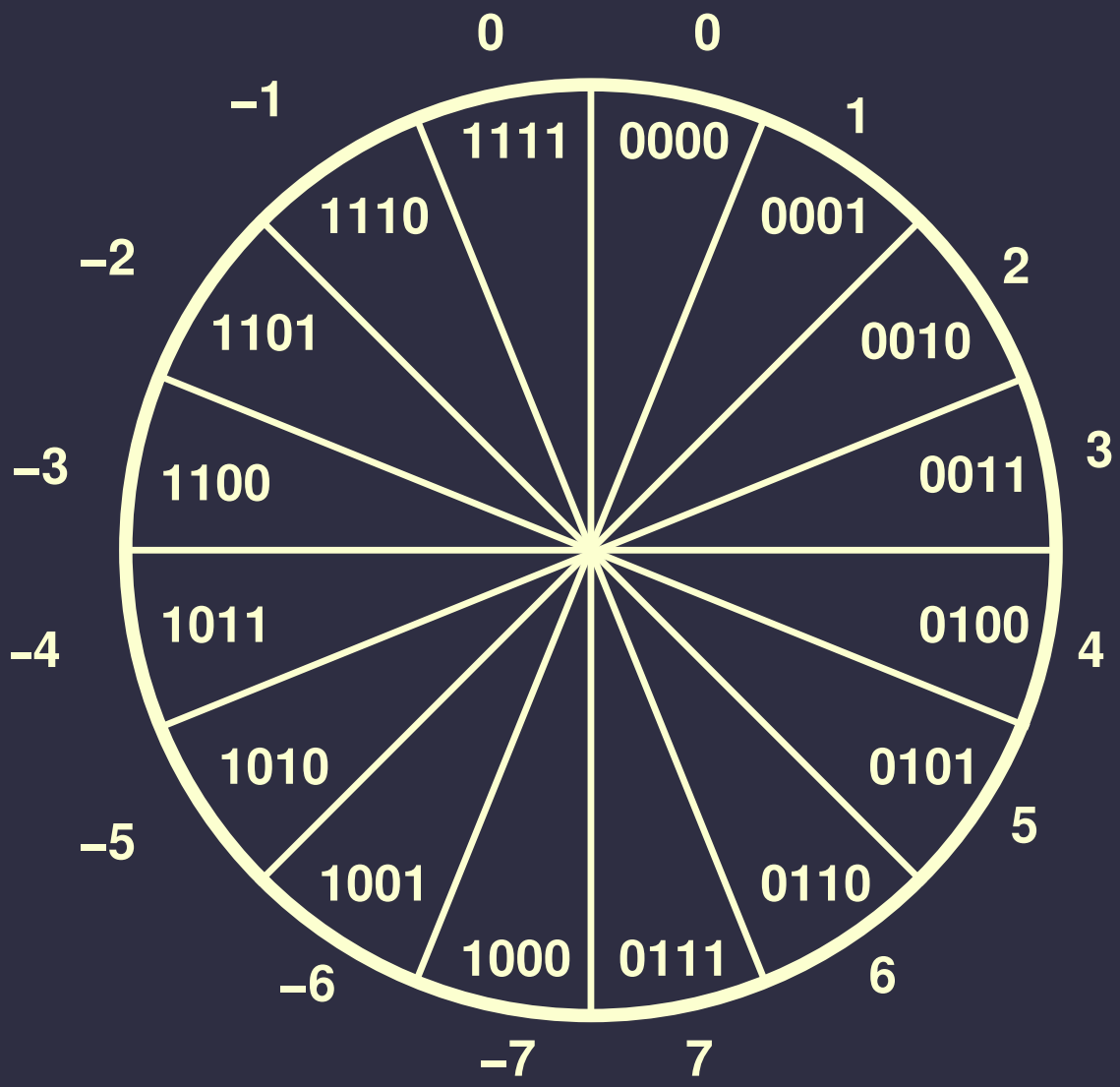
One's compliment



One's compliment

- If negative, complement all bits
- Addition somewhat simplified
- Do standard addition except wrap around carry back to the 0th bit
- Potentially requires two additions of the whole width
  - Slow

One's complement addition

Consider adding -5 (1010) and 7 (0111)

$$\begin{array}{rcccc} & 1 & 0 & 1 & 0 \\ + & 0 & 1 & 1 & 1 \\ \hline \end{array}$$

One's complement addition

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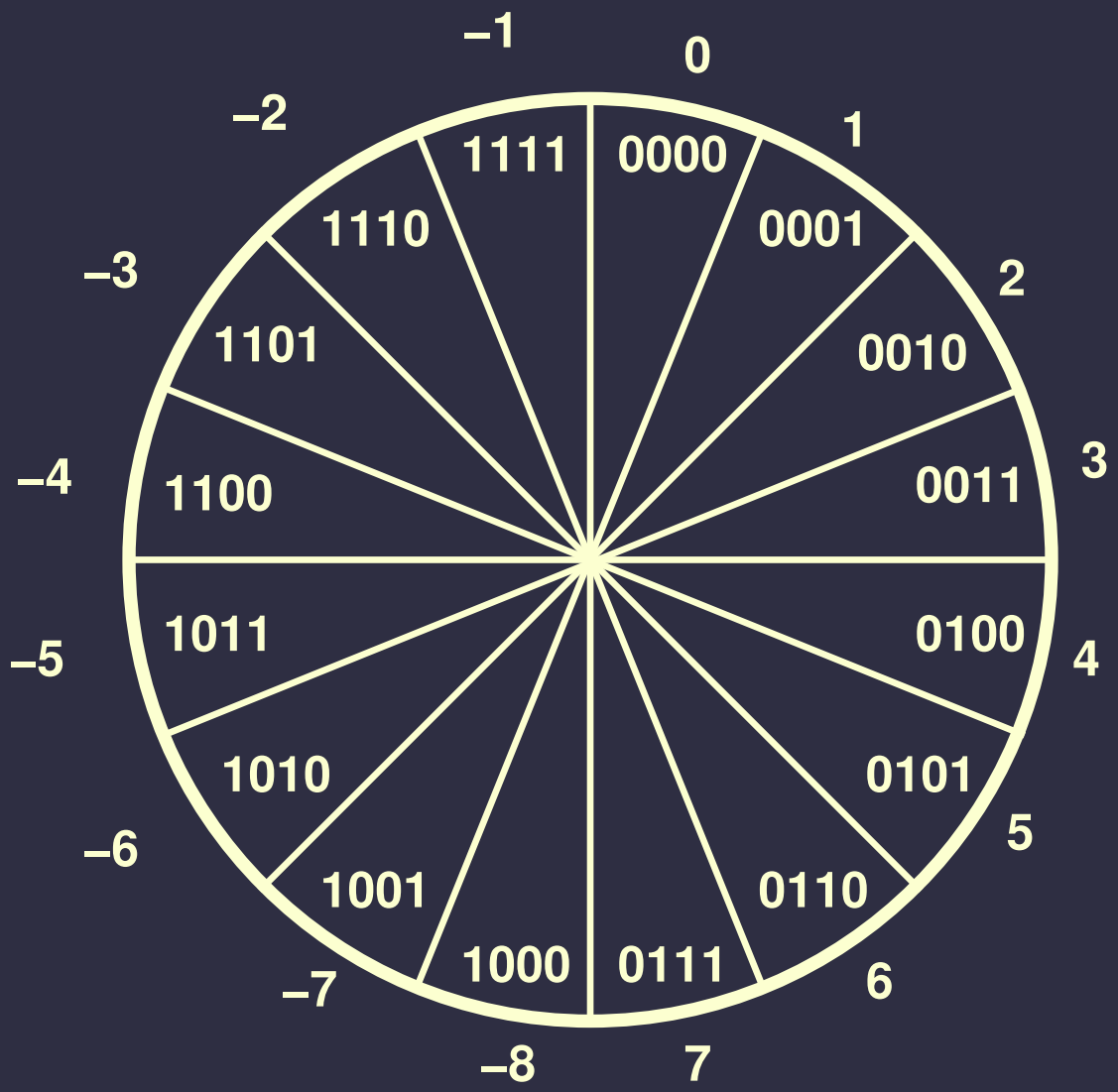
Two's complement

- To negate a number, invert all its bits and add 1
- Like one's complement, however, rotated by one bit
- Counter-intuitive
  - However, has some excellent properties

Two's complement



Two's complement

- Only one zero
  - Leads to more natural comparisons
- One more negative than positive number
  - This difference is irrelevant as n increases
- Substantial advantage – Addition is easy!

Two's complement addition

Consider adding -4 (1100) and 6 (0110)

$$\begin{array}{rcccc} & 1 & 1 & 0 & 0 \\ + & 0 & 1 & 1 & 0 \\ \hline \end{array}$$

Two's complement addition

Consider adding -4 (1100) and 6 (0110)

$$\begin{array}{rcccc} & 1 & 1 & 0 & 0 \\ + & 0 & 1 & 1 & 0 \\ \hline & & & & 0 \end{array}$$

Two's complement addition

Consider adding -4 (1100) and 6 (0110)

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Two's complement addition

Consider adding -4 (1100) and 6 (0110)

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Two's complement addition

Consider adding -4 (1100) and 6 (0110)

$$\begin{array}{cccccc} & 1 & 1 & & & \\ & & 1 & 1 & 0 & 0 \\ + & & 0 & 1 & 1 & 0 \\ \hline & & 0 & 0 & 1 & 0 \end{array}$$

Two's complement

- No looped carry – Only one addition necessary
- If carry-in to most-significant bit $\neq$ carry-out to most-significant bit, overflow occurs
- What does this represent?
- Both operands positive and have carry-in to sign bit
- Both operands negative and don't have carry-in to sign bit

Two's complement overflow

a	b	cin	cout
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Two's complement overflow

a	b	cin	cout
0	0	0	0
0	0	1	0
0	1	0	0
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1	0	0	0
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1. Addition and subtraction
   - Overview
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Half adder review

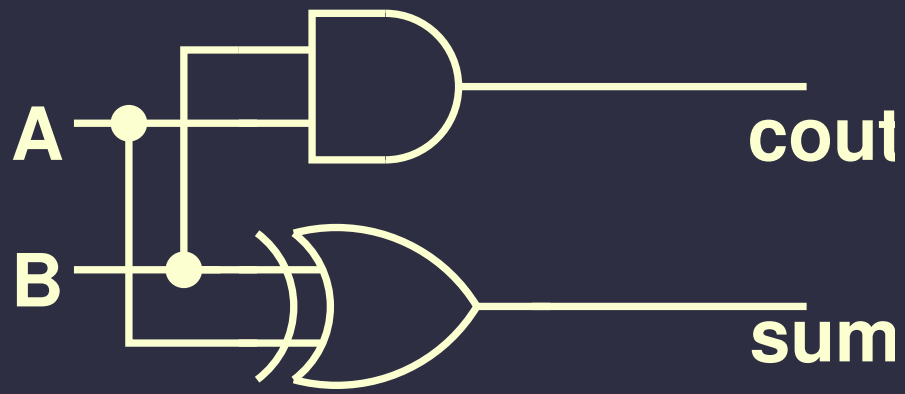
For two's complement, don't need subtracter

A	B	cout	sum
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

$$cout = AB$$

$$sum = A \oplus B$$

Half adder review



Full adder review

Need to deal with carry-in

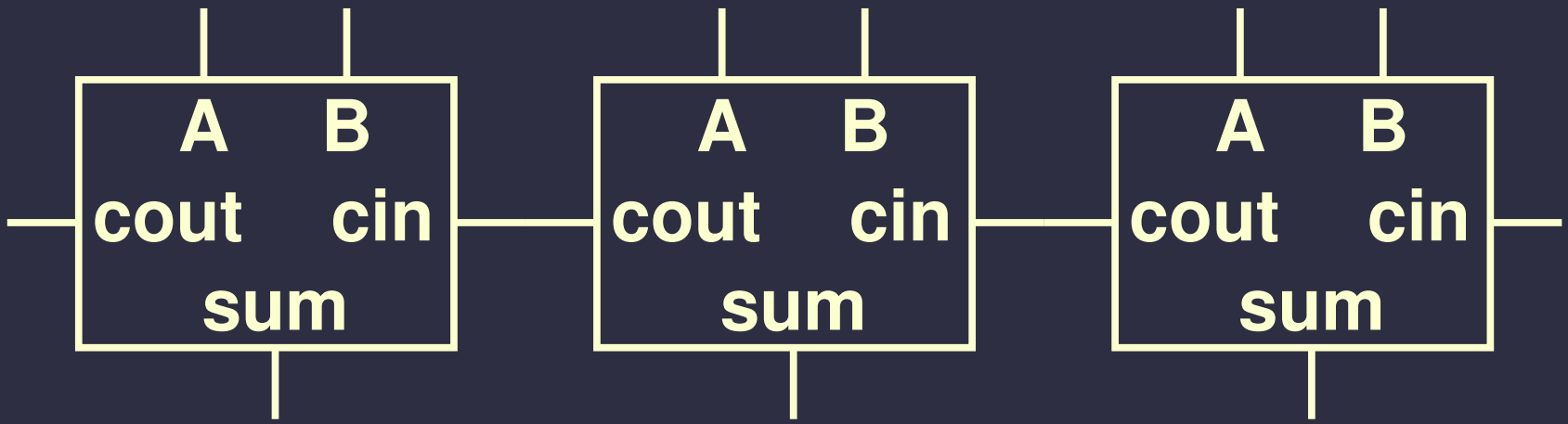
A	B	cin	cout	sum
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Full adder

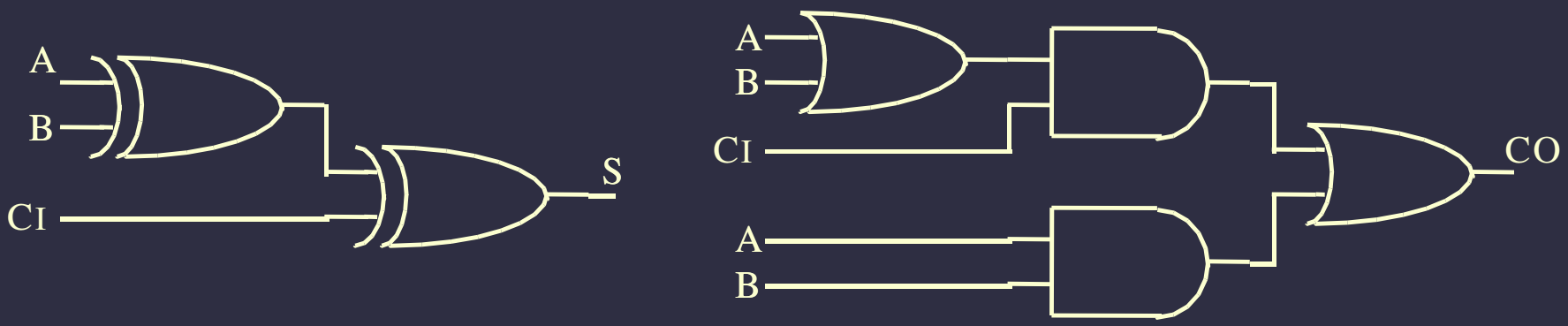
$$sum = A \oplus B \oplus cin$$

$$cout = AB + A\ ci + B\ ci$$

Cascaded full-adders

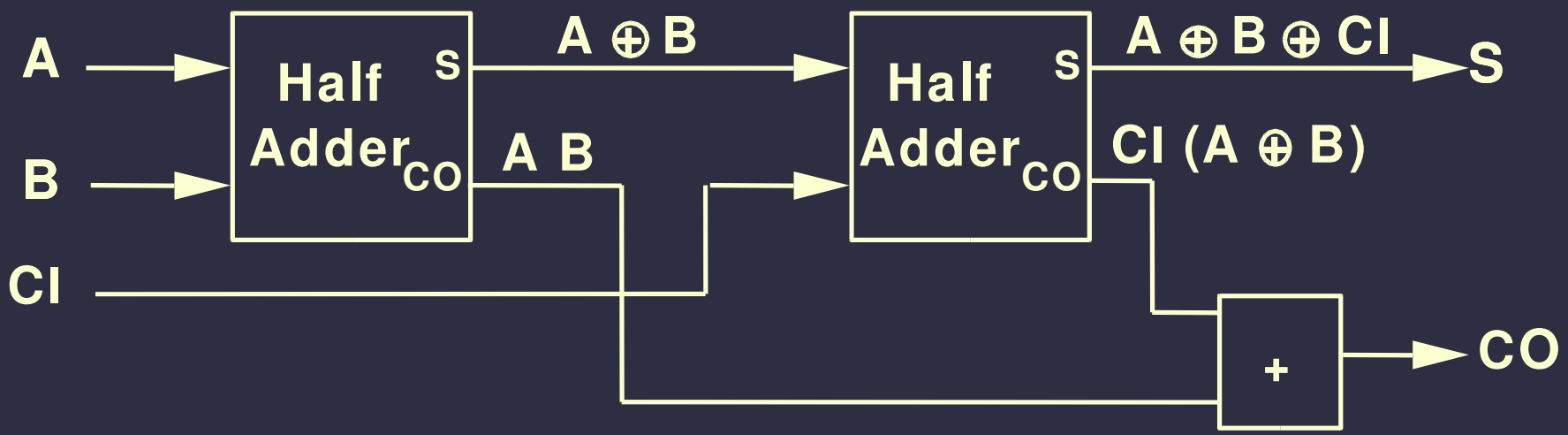


Full adder standard implementation



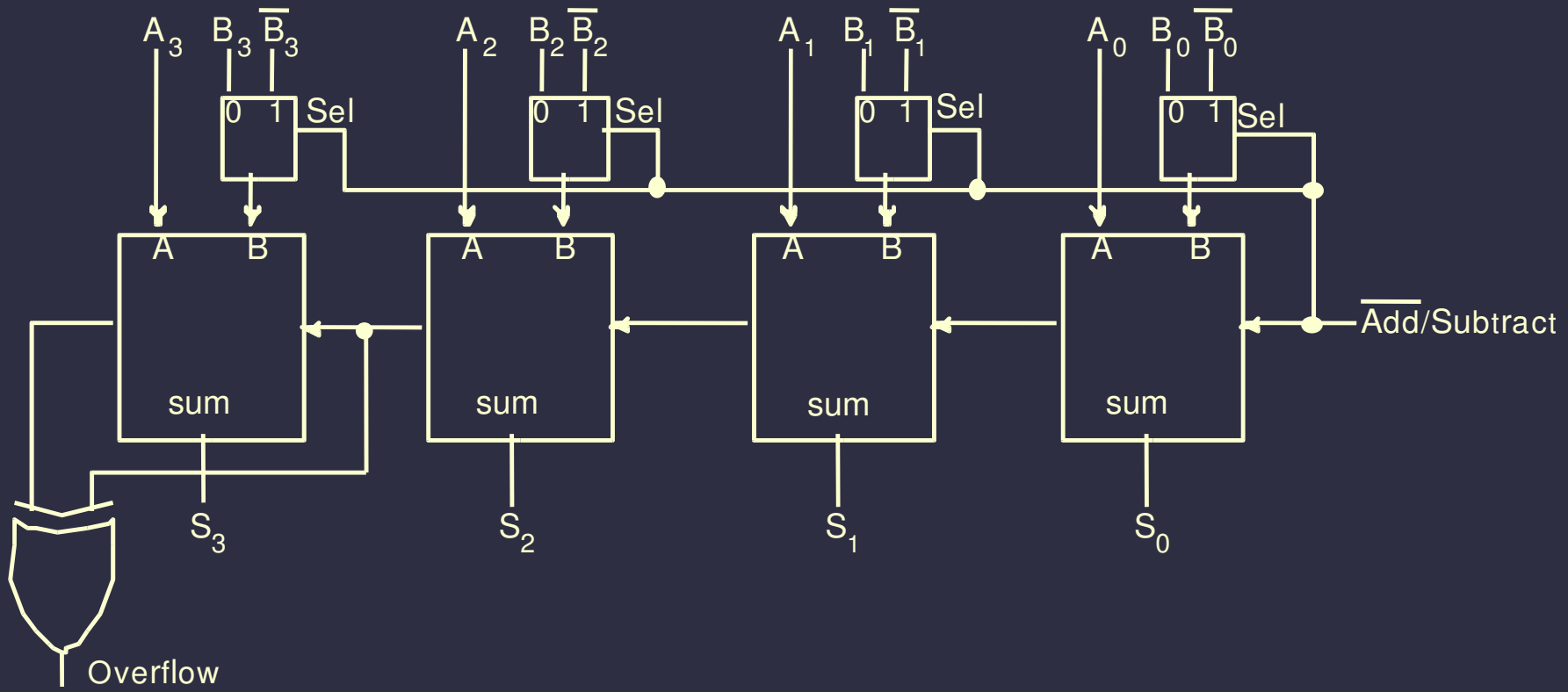
Six logic gates

Full adder composed of half-adders



$$AB + ci(A \oplus B) = AB + B\ ci + A\ ci$$

Adder/subtracter



Consider input to cin

Ripple-carry delay analysis

- The critical path (to *cout*) is two gate delays per stage
- Consider adding two 32-bit numbers
- 64 gate delays
  - Too slow!
- Consider faster alternatives

Carry lookahead adder

- Carry generate: $G = AB$
- Carry propagate: $P = A \oplus B$
- Represent *sum* and *cout* in terms of G and P

Carry lookahead adder

$$\begin{aligned} sum &= A \oplus B \oplus cin \\ &= P \oplus cin \end{aligned}$$

$$\begin{aligned} cout &= AB + A\ cin + B\ cin \\ &= AB + cin(A + B) \\ &= AB + cin(A \oplus B) \\ &= G + cin\ P \end{aligned}$$

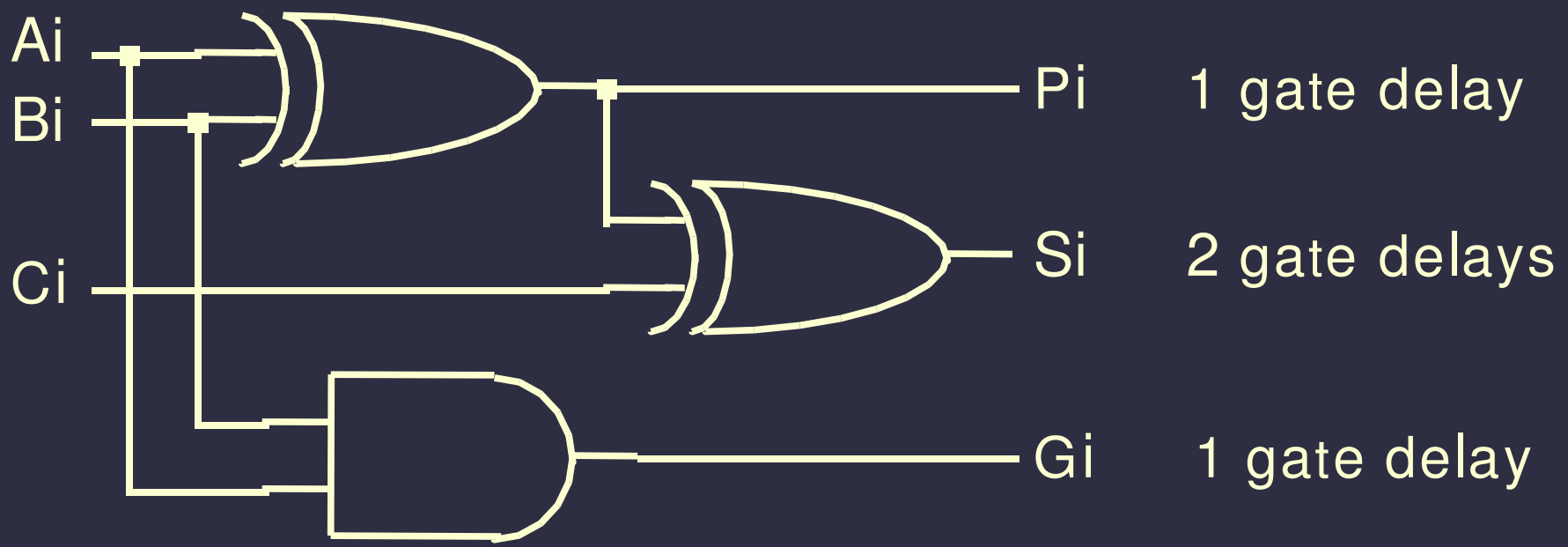
Carry lookahead adder

Flatten carry equations

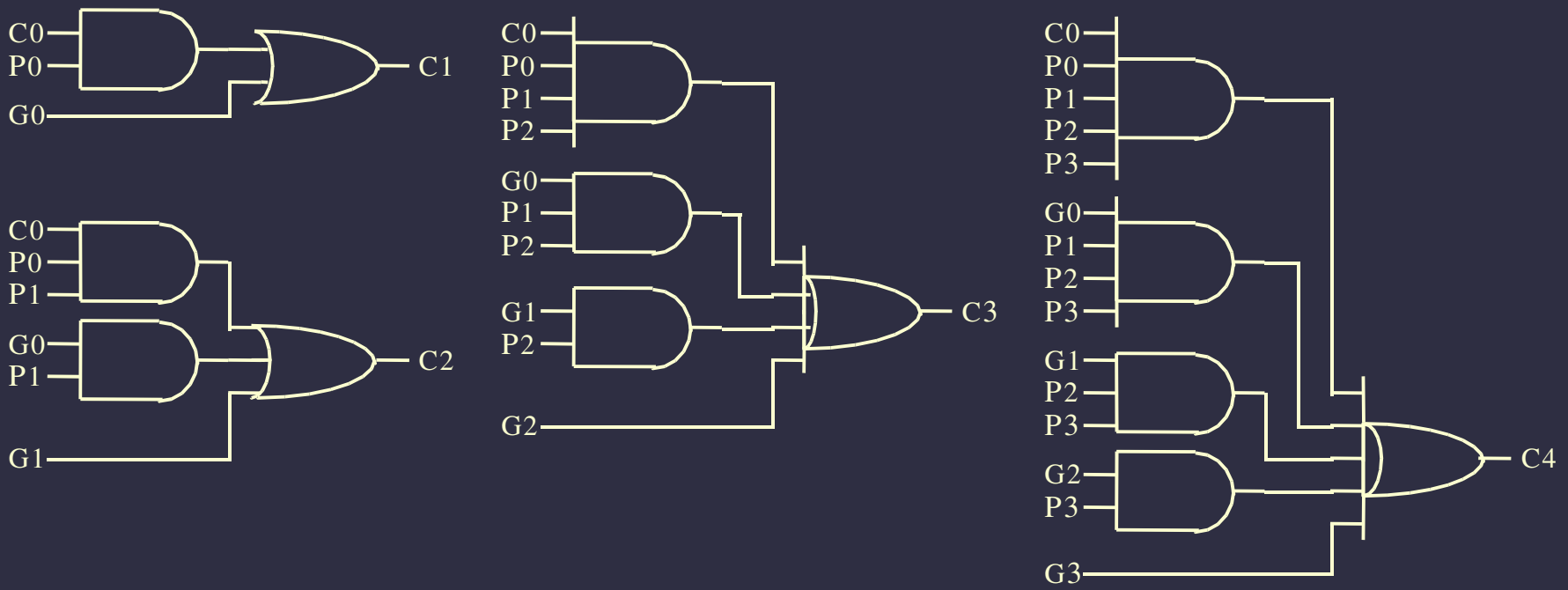
$$cin_1 = G_0 + P_0\ cin_0$$
$$cin_2 = G_1 + P_1\ cin_1 = G_1 + P_1 G_0 + P_1 P_0\ cin_0$$
$$cin_3 = G_2 + P_2\ cin_2 = G_2 + P_2 G_1 + P_2 P_1 G_0 + P_2 P_1 P_0\ cin_0$$
$$cin_4 = G_3 + P_3 C_3 = G_3 + P_3 G_2 + P_3 P_2 G_1 +$$
$$P_3 P_2 P_1 G_0 + P_3 P_2 P_1 P_0\ cin_0$$

Each *cin* can be implemented in three-level logic

Carry lookahead building block



Carry lookahead adder



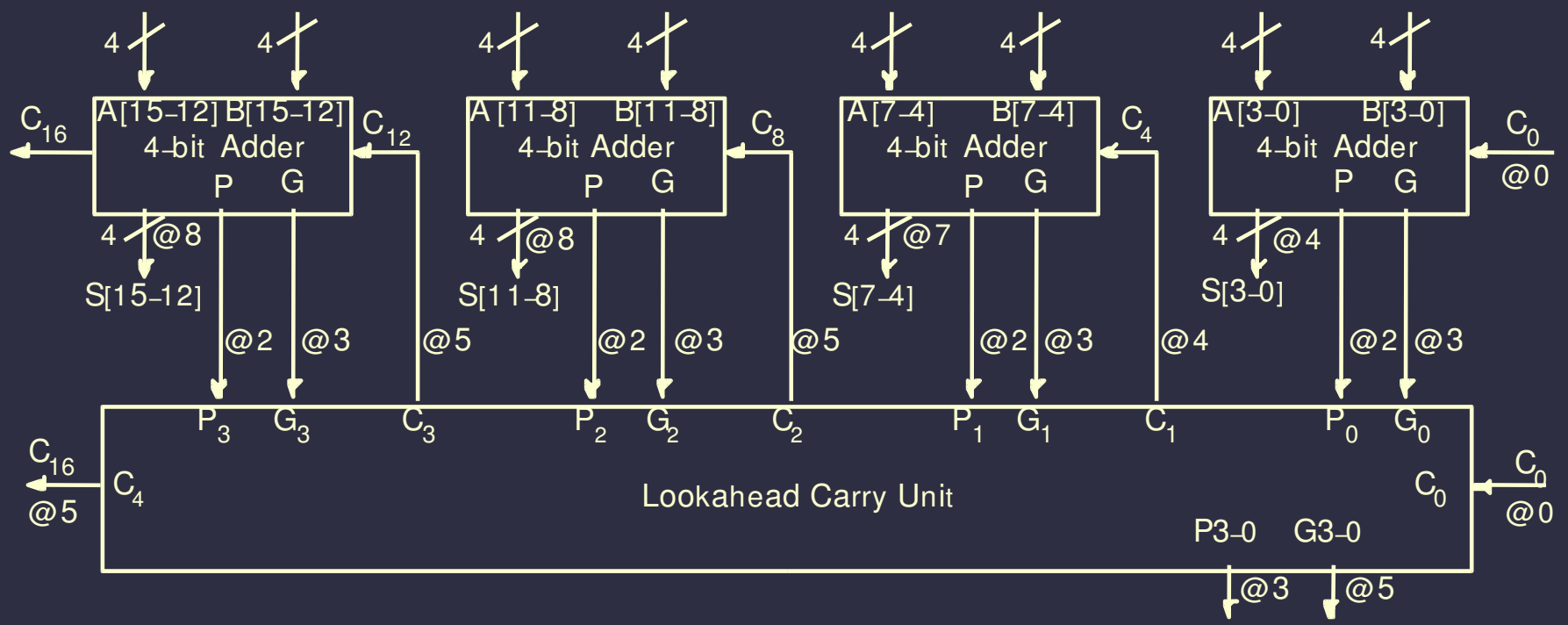
Carry lookahead delay analysis

- Assume a 4-stage adder with CLA
- Propagate and generate signals available after 1 gate delays
- Carry signals for slices 1 to 4 available after 3 gate delays
- Sum signal for slices 1 to 4 after 4 gate delays

Carry lookahead

- No carry chain slowing down computation of most-significant bit
  - Computation in parallel
- More area required
- Each bit has more complicated logic than the last
- Therefore, limited bit width for this type of adder
- Can chain multiple carry lookahead adders to do wide additions
- Note that even this chain can be accelerated with lookahead
  - Use internal and external carry lookahead units

Cascaded carry lookahead adder



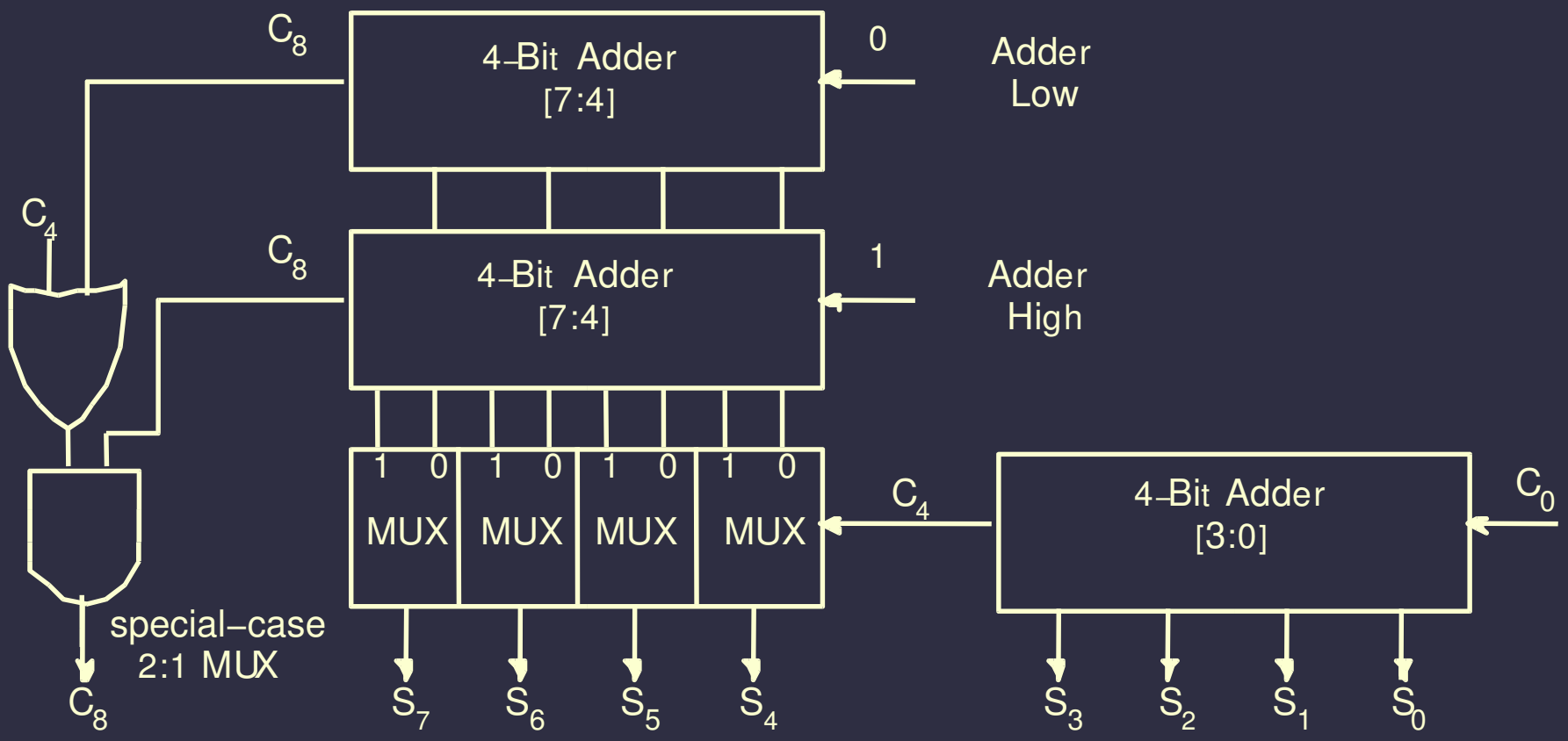
Delay analysis for cascaded carry lookahead

- Four-stage 16-bit adder
- *cin* for MSB available after five gate delays
- *sum* for MSB available after eight gate delays
- 16-bit ripple-carry adder takes 32 gate delays
- Note that not all gate delays are equivalent
- Depends on wiring, driven load
- However, carry lookahead is usually much faster than ripple-carry

Carry select adders

- Trade even more hardware for faster carry propagation
- Break a ripple carry adder into two chunks, *low* and *high*
- Implement two *high* versions
  - $high_0$ computes the result if the carry-out from *low* is 0
  - $high_1$ computes the result if the carry-out from *low* is 1
- Use a MUX to select a result once the carry-out of *low* is known
  - $high_0$'s *cout* is never greater than $high_1$'s *cout* so special-case MUX can be used

Carry select adder



Delay analysis of carry select adder

- Consider 8-bit adder divided into 4-bit stages
- Each 4-bit stage uses carry lookahead
- The 2:1 MUX adds two gate delays
- 8-bit sum computed after 6 gate delays
- 7 gate delays for carry lookahead
- 16 gate delays for ripple carry

Outline

Arithmetic operations

- Digital logic circuits frequently need to carry out arithmetic operations
  - Addition, subtraction, and multiplication
- A number of design decisions affect the performance, area, and power consumption of arithmetic sub-circuits
- Number systems
- Trade-off between area/power consumption and speed

Multiplication

- To understand why these trade-offs exist, we need to understand the fundamentals of arithmetic circuits
- We have already discussed the selection of number systems and the design of adders/subtracters
- Similar alternatives exist for multipliers

Multiplication

- Multiplication is the repeated application addition of ANDed bits and shifting (multiplying by two)
- Multiplication is the sum of the products of each bit of one operand with the other operand
- Consequence: A product has double the width of its operands

Multiplication

Recall that multiplying a number by two shifts it to the left one bit

$$\begin{aligned} 6 \cdot 3 &= 6 \cdot (2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1) \\ &= 6 \cdot 2^2 \cdot 0 + 6 \cdot 2^1 \cdot 1 + 6 \cdot 2^0 \cdot 1 \\ 110 \cdot 011 &= 11000 \cdot 0 + 1100 \cdot 1 + 110 \cdot 1 \\ &= 110 + 1100 \\ &= 10010 \\ &= 18 \end{aligned}$$

Multiplication

$$
\begin{array}{cccccccc}
 & & & & A_2 & A_1 & A_0 \\
 & & & \times & B_2 & B_1 & B_0 \\
\hline
 & & & & A_2B_0 & A_1B_0 & A_0B_0 \\
 & & & A_2B_1 & A_1B_1 & A_0B_1 & \\
+ & & A_2B_2 & A_1B_2 & A_0B_2 & & \\
\hline
 & sum_5 & sum_4 & sum_3 & sum_2 & sum_1 & sum_0
\end{array}
$$

Multiplication

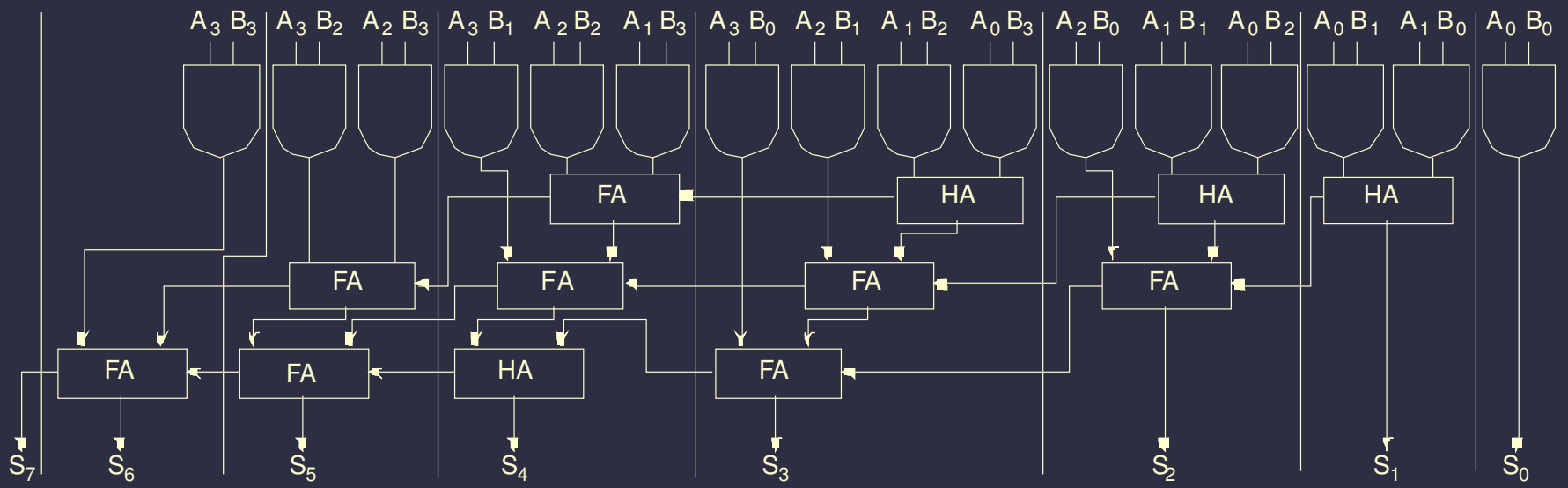
Consider multiplying 6 (110) by 3 (011)

$$
\begin{array}{cccccc}
 & & & 1 & 1 & 0 \\
 & & \times & 0 & 1 & 1 \\
\hline
 & & & 1 & 1 & 0 \\
 & & 1 & 1 & 0 & \\
+ & 0 & 0 & 0 & & \\
\hline
0 & 1 & 0 & 0 & 1 & 0
\end{array}
$$

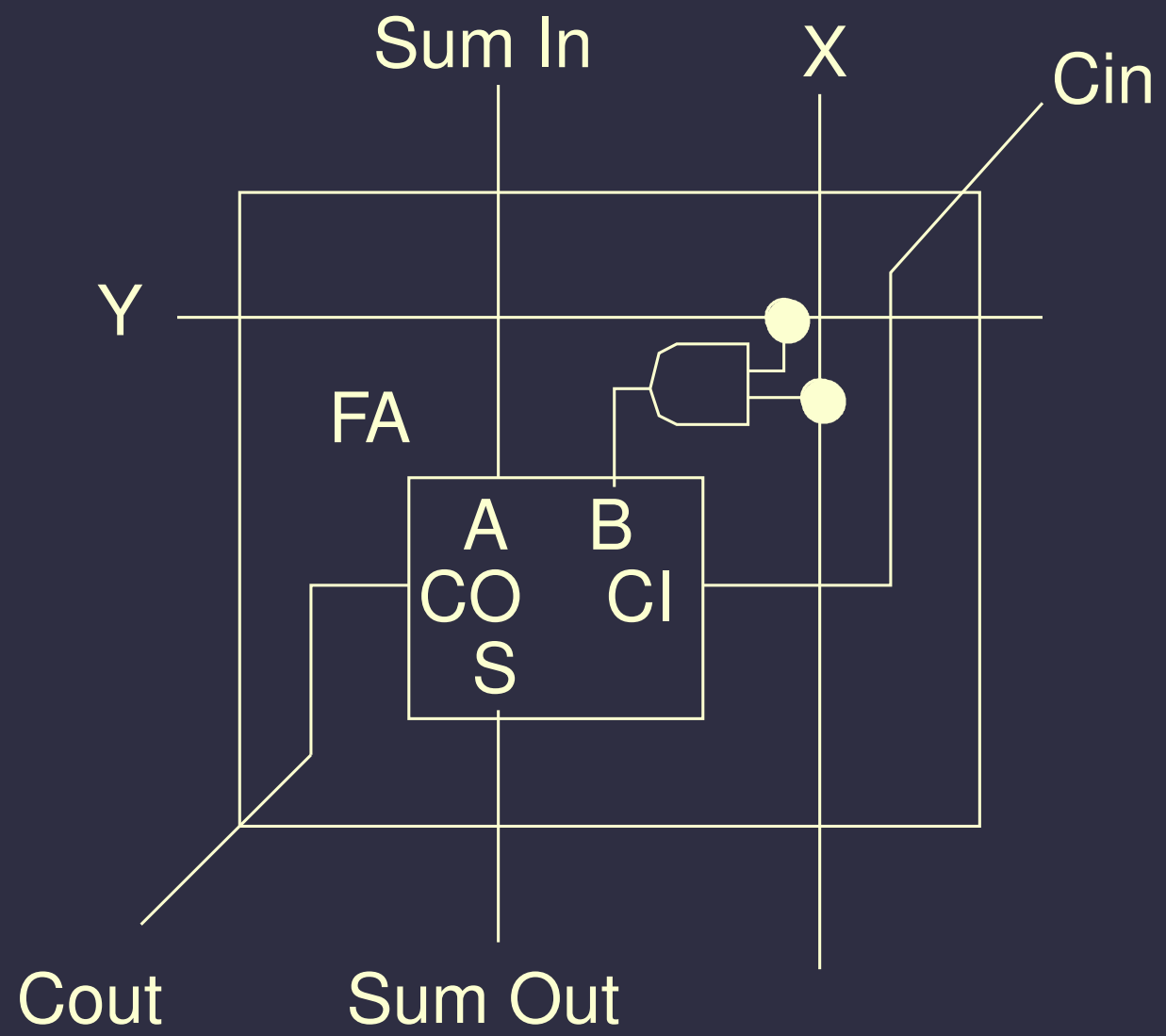
Multiplier implementation

- Direct implementation of this scheme possible
- Partial products formed with ANDs
- For four bits, 12 adders and 16 gates to form the partial products
  - 88 gates
- Note that the maximum height (number of added bits) is equal to the operand width

Combinational multiplier



Multiplier building block



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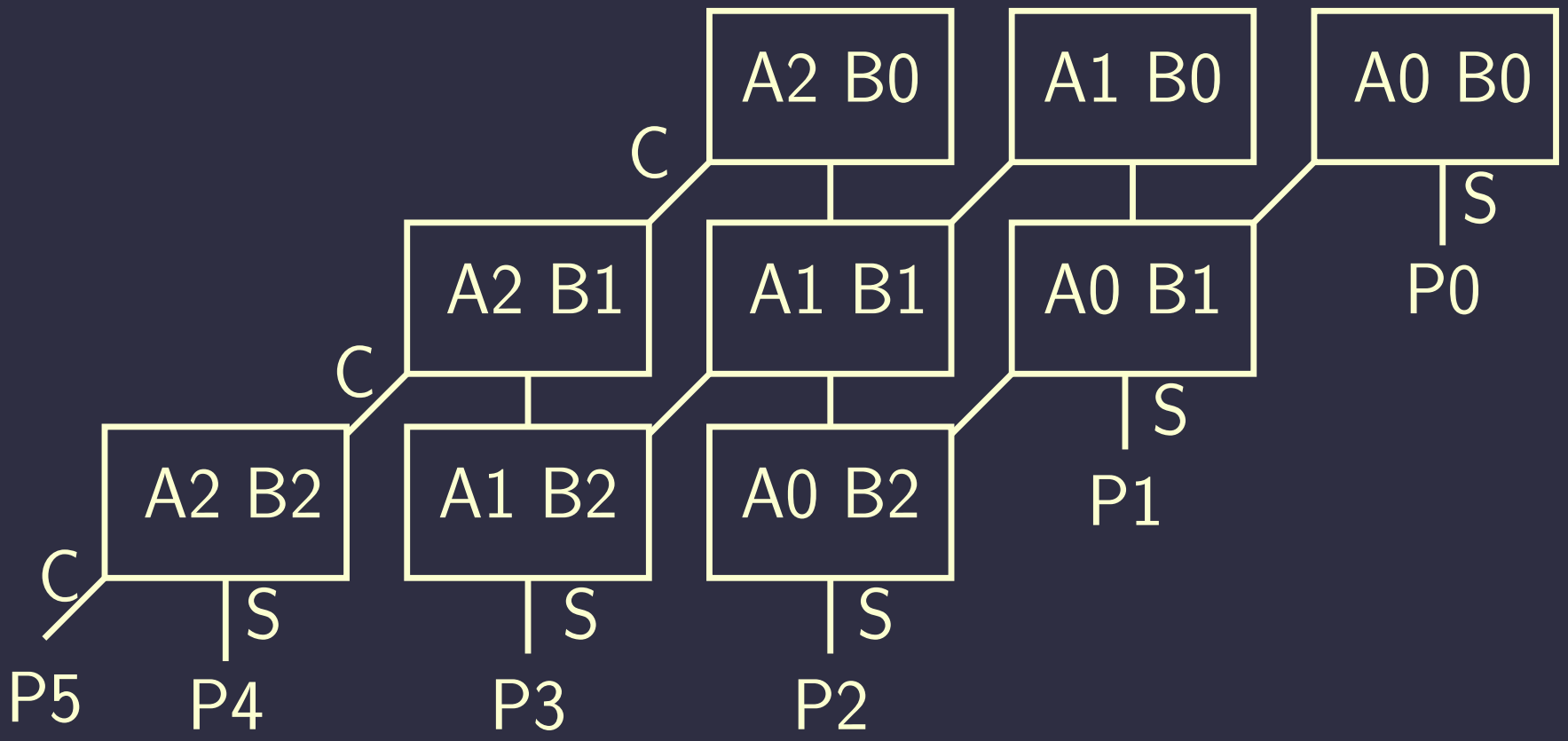
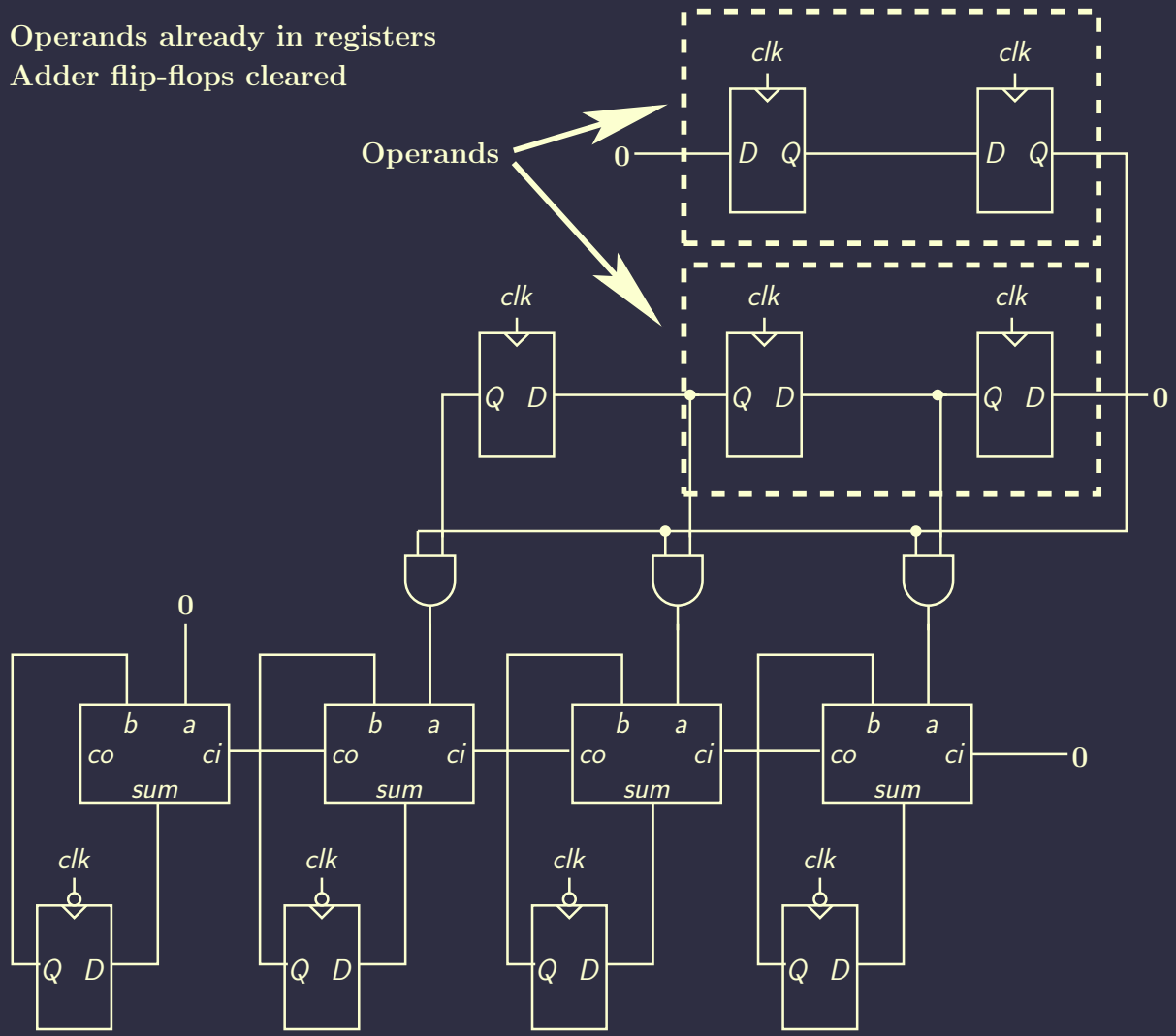


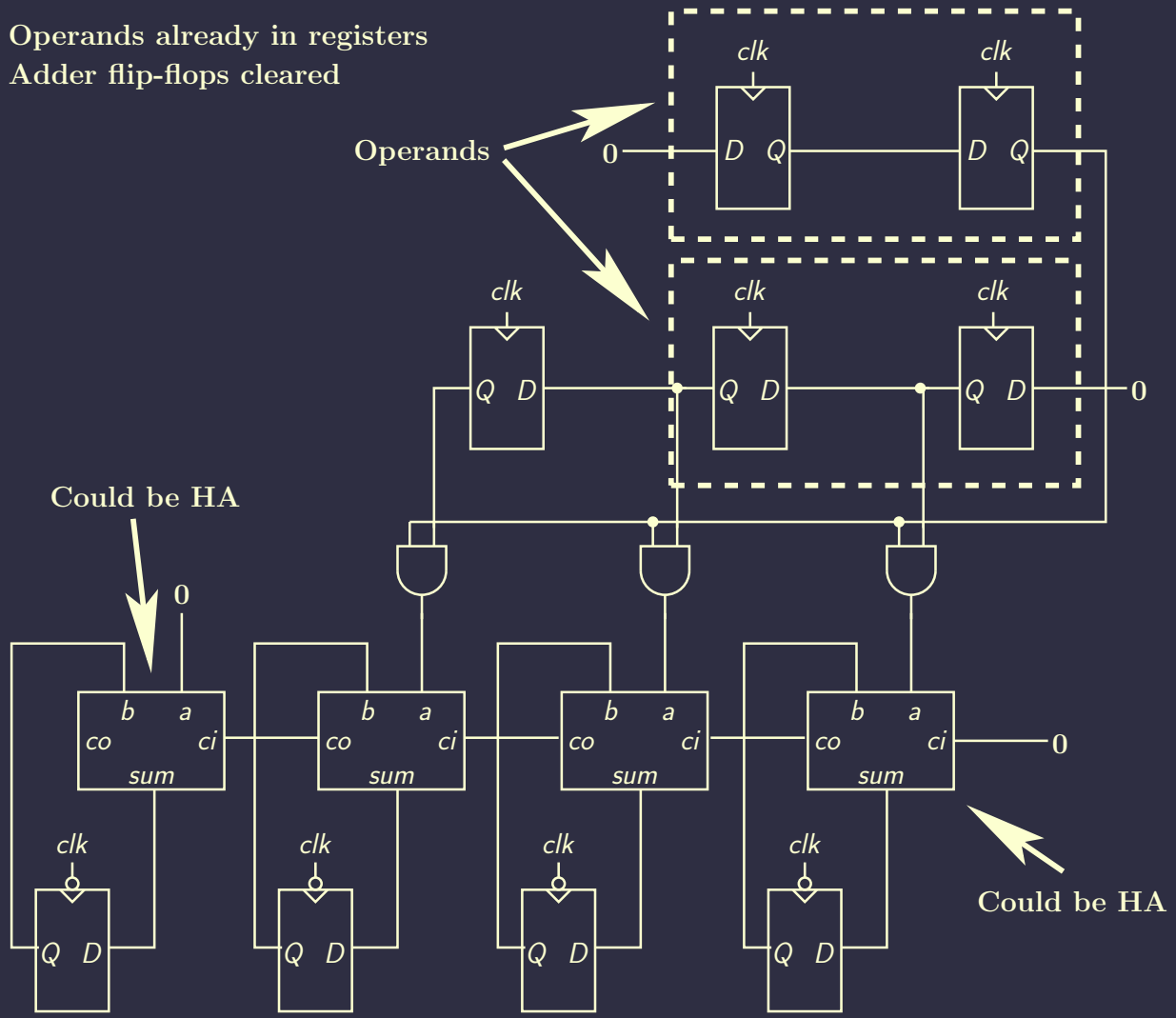
Sequential multiplier

- Can iteratively one row of adders to carry out multiplications
- Advantage: Area reduced to approximately its square root
- Disadvantage: Takes n clock cycles, where n is the operand bit width

2X2 sequential multiplier



2X2 sequential multiplier



Arithmetic/logic units

- Possible to implement functional units that can carry out many arithmetic and logic operations with little additional area or delay overhead
- Already saw example: Combined adder/subtracter
- Other operations possible
- Could you generalize the approach used for two's compliment addition and subtraction to another pair of operations?

Arithmetic/logic operations

- Increment
- Addition
- Negation
- Subtraction
- Multiplication
  - Slow or large
- Division
  - Slow or large

Arithmetic/logic operations

- Shift left
  - Fast, multiplication by two
- Shift right
  - Fast, division by two
- Bit-wise operations
  - AND, OR, NOT, NAND, NOR, XOR, and XNOR

Memory types

- ROM, PROM, EPROM, EEPROM: Already know these
- SRAM: Fast, low-density, relies on feedback
- DRAM: Fast, high-density, requires refresh, relies on stored charge
- Flash: Already know these – Non-volatile, slow, relies on floating gate

Outline

Recommended reading

- M. Morris Mano and Charles R. Kime. *Logic and Computer Design Fundamentals*. Prentice-Hall, NJ, third edition, 2004
- Chapter 9